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Abstract: In Operational Modal Analysis, the modal parameters (natural frequencies, damping ratios and mode shapes), obtained from Stochastic System Identification of structures, are subject to statistical uncertainty from ambient vibration measurements. It is hence necessary to evaluate the confidence intervals of these obtained results. This paper will propose an algorithm that can efficiently estimate the uncertainty on modal parameters obtained from the Eigensystem-Realization-Algorithm (ERA).

Key-words: Operational Modal Analysis, Stochastic System Identification, Error Quantification, Mechanical and Aerospace

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Calculs d'incertitude pour la méthode d'identification ERA

Résumé : En Analyse modale, les paramètres modaux (fréquences, amortissements et déformées), obtenus à partir des methodes stochastiques sont sujet à des erreurs d'incertitude. Il est nécessaire d'évaluer les intervalles de confiance correspondants. Ce calcul est fait pour une classe d'algorithmes sous espaces appelés ERA (Eigensystem-Realization-Algorithm).

Mots-clés : Analyse modale, identification des systèmes stochastiques, calcul d'incertitudes, mécanique et aerospatiale

1 Introduction

The design and maintenance of mechanical structures subject to noise and vibrations is an important topic in mechanical engineering. It is an important component of comfort (cars and buildings) and contributes signicantly to the safety related aspects of design and maintenance (aircrafts, aerospace vehicles and payloads, civil structures). Requirements from these application areas are numerous and demanding. Laboratory and in-operation tests are performed on the prototype structure, in order to get so-called modal models, i.e., to extract the modes and damping factors (these correspond to system poles), the mode shapes (corresponding eigenvectors), and loads. These results are used for updating the design model for a better fit to data, and sometimes for certification purposes (e.g., in flight domain opening for new aircrafts).

The estimation of modal parameters of structures can easily be carried out by using Stochastic System Identification methods on sensor measurements. [3] proved that the Instrumental Variable method and what was called the Balanced Realization method for linear eigenstructure identification are consistent in a nonstationary context. From that on, the family of subspace algorithms has been extensively studied (see in [9, 14]) and has expanded rapidly. There are a number of convergence studies on subspace methods in the literature (see [6, 2, 1, 5]) to mention just a few of them. These papers provide deep and technically difficult results including convergence rates. Our objective is to derive simple formula for such sensitivities. Sensitivities for the algorithms considered in this paper, ERA, are not addressed by those papers.

The uncertainty on modal parameters appears for many reasons, e.g. finite number of data samples, undefined measurement noises, nonstationary excitations, nonlinear structure, model order reduction,..., then the system identification algorithms do not yield the exact system matrices. Practically, the statistical uncertainty of the obtained modal parameters at a chosen system order can be computed from the uncertainty of the system matrices, which depends on the covariance of the corresponding subspace matrix. Not knowing the model order yields to use empirical multi-order procedure such as the stabilization diagram ([10]), where modes of the system are assumed to stabilize when the model order increases.

In [12], it has been shown how confidence intervals of modal parameters can be determined from the covariances of the system matrices and the covariances of subspace matrices. The current paper will expand on this and compare sensitivities for two output-only system identification methods, namely output-only Stochastic Subspace Identification (SSI) and Eigensystem-Realization-Algorithm (ERA) System Identification. Subspace identification is based on the computation of one subspace matrix from the correlation tail. Unlike subspace algorithm, ERA computes the system matrices using the information of both (k)- and (k+1)-lag of shifted correlation tails.

In this paper, following the lines of ([12]), an algorithm will be developed for estimating the confidence intervals in ERA system identification. The uncertainty on state transition matrix is derived, based on the uncertainties of (k)- and (k+1)-lag subspace matrices.

A relevant industrial example is applied to ERA estimates. The efficiency of these algorithms and lag effect are also taken into account. Comparison with subspace algorithm estimates is also performed.

2 Stochastic System Identification

2.1 The General SSI Algorithm

The discrete time model in state-space form is:

$$\begin{cases} X_{k+1} & = & AX_k + V_{k+1} \\ Y_k & = & CX_k \end{cases} \tag{1}$$

with the state $X \in \mathbb{R}^n$, the output $Y \in \mathbb{R}^r$, the state transition matrix $A \in \mathbb{R}^{n \times n}$ and the observation matrix $C \in \mathbb{R}^{r \times n}$. The state noise V is unmeasured and assumed to be Gaussian, zero-mean, white.

Let r be the number of sensors, p and q be chosen parameters with $(p+1)r \geq qr \geq n$. From the output data, a matrix $\mathcal{H}_{p+1,q} \in \mathbb{R}^{(p+1)r \times qr}$ is built according to a chosen SSI algorithm, see e.g. [4] for an overview. The matrix $\mathcal{H}_{p+1,q}$ will be called "subspace matrix" in the following, and the SSI algorithm is chosen such that the corresponding subspace matrix enjoys (asymptotically for a large number of samples) the factorization property

$$\mathcal{H}_{p+1,q} = \mathcal{O}_{p+1} \ \mathcal{Z}_q \tag{2}$$

into the matrix of observability

$$\mathcal{O}_{p+1} \stackrel{\text{def}}{=} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^p \end{bmatrix} \tag{3}$$

and a matrix $\mathcal{Z}_q$ depending on the selected SSI algorithm.

Let N be the number of available samples and $Y_k \in \mathbb{R}^r$, $\{k \in 1, \dots, N\}$ the vector containing the sensor data. Then, the "forward" and "backward" data matrices

$$\begin{aligned} \mathcal{Y}_{p+1}^{+} &= \frac{1}{\sqrt{N-p-q}} \begin{bmatrix} Y_{q+1} & Y_{q+2} & \dots & Y_{N-p} \\ Y_{q+2} & Y_{q+3} & \dots & Y_{N-p+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{q+p+1} & Y_{q+p+2} & \dots & Y_N \end{bmatrix}, \\ \mathcal{Y}_q^{-} &= \frac{1}{\sqrt{N-p-q}} \begin{bmatrix} Y_q & Y_{q+1} & \dots & Y_{N-p-1} \\ Y_{q-1} & Y_q & \dots & Y_{N-p-2} \\ \vdots & \vdots & \ddots & \vdots \\ Y_1 & Y_2 & \dots & Y_{N-p-q} \end{bmatrix} \end{aligned} \tag{4}$$

are built. For the *covariance-driven* SSI (see also [3], [10]), the subspace matrix $\mathcal{H}_{p+1,q}^{(\text{cov})} = \mathcal{Y}_{p+1}^{+} \mathcal{Y}_q^{-T}$ is built, which enjoys the factorization property (2), where $\mathcal{Z}_q$ is the controllability matrix.

For simplicity, let p and q be given, skip the subscripts of $\mathcal{H}_{p+1,q}$, $\mathcal{O}_{p+1}$ and $\mathcal{Z}_q$. The eigenstructure of the system (1) is retrieved from a given matrix $\mathcal{H}$.

The observability matrix $\mathcal{O}$ is obtained from a thin Singular Value Decomposition (SVD) of the matrix $\mathcal{H}$ and its truncation at the desired model order

n:

$$\begin{aligned}\mathcal{H} &= U\Sigma V^T \\ &= \begin{bmatrix} U_1 & U_0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{bmatrix} V^T, \quad (5)\\ \mathcal{O} &= U_1 \Sigma_1^{1/2}. \quad (6)\end{aligned}$$

Note that the singular values in $\Sigma_1 \in \mathbb{R}^{d\times d}$ must be non-zero and hence $\mathcal{O}$ is of full column rank. The observation matrix C is then found in the first block-row of the observability matrix $\mathcal{O}$. The state transition matrix A is obtained from the shifting invariance property of $\mathcal{O}$, namely as the least squares solution of

$$\mathcal{O}\!\uparrow\! A = \mathcal{O}\!\downarrow, \text{ where } \mathcal{O}\!\uparrow \stackrel{\text{def}}{=} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}, \ \mathcal{O}\!\downarrow \stackrel{\text{def}}{=} \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^p \end{bmatrix}. \quad (7)$$

The eigenstructure $(\lambda, \varphi_\lambda)$ results from

$$\det(A - \lambda I) = 0, \quad A\phi_\lambda = \lambda \phi_\lambda, \quad \varphi_\lambda = C\phi_\lambda, \quad (8)$$

where λ ranges over the set of eigenvalues of A. From λ, the natural frequency and damping ratio are obtained, and φ_λ is the corresponding mode shape.

There are many papers on the used identification techniques. A complete description can be found in [3], [14], [10], [4], and the related references. A proof of non-stationary consistency of these subspace methods can be found in [4].

2.2 ERA (Eigensystem-Realization-Algorithm)

Another variant of realization algorithm based on the computation of the subspace matrices is called ERA (Eigensystem-Realization-Algorithm) (see in [8]). It is based on the general remark that one can compute the subspace matrix $\mathcal{H}$ not using the first lags of the correlation tail. Defining $\mathcal{H}^{(k)}$ as

$$\mathcal{H}^{(k)} = \begin{bmatrix} R_{k+1} & R_{k+2} & \dots & R_{k+q} \\ R_{k+2} & R_{k+3} & \dots & R_{k+q+1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{k+p+1} & \dots & \dots & R_{k+p+q} \end{bmatrix}, \quad (9)$$

in which the correlations are related to the factorization

$$R_j \stackrel{\text{def}}{=} \mathbf{E}\left(Y_{l+j}\, Y_l^T\right) = CA^j G \quad (10)$$

with the cross-covariance between the state and the observed outputs $G = \mathbf{E}\,[X_l\, Y_l^T]$.

Then, a Singular Value Decomposition is performed on $\mathcal{H}^{(k)}$ as

$$\mathcal{H}^{(k)} = \begin{bmatrix} U_1 & U_0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_0^T \end{bmatrix} \quad (11)$$

The state transition matrix will be defined as

$$A = \left(\mathcal{O}_1^\dagger\right) \mathcal{H}^{(k+1)} \left(\mathcal{Z}_1^\dagger\right), \quad (12)$$

where † means Moore-Penrose pseudo-inverse, and

$$\mathcal{O}_1^\dagger = (\Sigma_1)^{-\frac{1}{2}} U_1^T, \tag{13}$$

$$\mathcal{Z}_1^\dagger = V_1 (\Sigma_1)^{-\frac{1}{2}}. \tag{14}$$

If the correlations are computed from cross spectra, the method is called NEXT-ERA; but without loss of generality, it is just assumed that the correlations are computed from time samples. The dimensions of A relates to the dimensions of U_1, Σ_1, V_1. And as such, a stabilization diagram is obtained by performing the computation of A for multiple model orders and keeping as stable poles the modes which repeat over multiple model orders.

3 Confidence intervals

3.1 Descriptions of SSI Confidence Intervals algorithm

The statistical uncertainty of the obtained modal parameters at a chosen system order can be computed from the uncertainty of the system matrices, which depends on the covariance of the corresponding subspace matrix $\mathcal{H}$. The latter can be evaluated by cutting the sensor data into blocks on which instances of the subspace matrix are computed. So, this offers a possibility to compute the confidence intervals of the modal parameters at a certain system order without repeating the system identification. In [12], this algorithm was described in detail for the covariance-driven SSI. The uncertainty ΔA and ΔC of the system matrices A and C are connected to the uncertainty of the subspace matrix through a Jacobian matrix

$$\begin{bmatrix} vec\Delta A \\ vec\Delta C \end{bmatrix} = J_{A,C}\ vec\Delta\mathcal{H}, \tag{15}$$

where vec is the vectorization operator. Then, the uncertainty of the modal parameters (natural frequency f, damping ratio d and mode shape ϕ) is derived from

$$\Delta f_\mu = J_{f_\mu} \begin{bmatrix} vec\Delta A \\ vec\Delta C \end{bmatrix}, \quad \Delta d_\mu = J_{d_\mu} \begin{bmatrix} vec\Delta A \\ vec\Delta C \end{bmatrix}, \tag{16}$$

and

$$\Delta \phi_\mu = J_{\phi_\mu} \begin{bmatrix} vec\Delta A \\ vec\Delta C \end{bmatrix}. \tag{17}$$

The Jacobians J_{f_μ}, J_{d_μ} and J_{ϕ_μ} are computed for each mode μ. Finally, the covariances of the modal parameters are obtained as

$$\begin{aligned} cov(f_\mu) &= J_{f_\mu}\ J_{A,C}\ cov(vec\ \mathcal{H})\ J_{A,C}^T\ J_{f_\mu}^T \\ cov(d_\mu) &= J_{d_\mu}\ J_{A,C}\ cov(vec\ \mathcal{H})\ J_{A,C}^T\ J_{d_\mu}^T \\ cov(\phi_\mu) &= J_{\phi_\mu}\ J_{A,C}\ cov(vec\ \mathcal{H})\ J_{A,C}^T\ J_{\phi_\mu}^T \end{aligned} \tag{18}$$

where $cov(vec\mathcal{H})$ is the covariance of the vectorized subspace matrix. After retrieving the uncertainties on the system matrices A and C, the calculation of the uncertainties on the frequency and damping is straightforward. However, for the mode shape, there is an issue of normalization as each one is defined up to an unknown constant. This was addressed in [7].

3.2 Derivation of ERA Confidence Intervals

In this section, for ERA, it is investigated how the covariances of modal parameters can be derived from the covariance of subspace matrices taking care of the uncertainties of observability, controllability and system matrices.

Firstly, the uncertainty on the system matrix A is a function of the sensitivities of $\mathcal{H}^{(k+1)}$, $\mathcal{O}_1^\dagger$ and $\mathcal{Z}_1^\dagger$:

$$\begin{aligned} \Delta A &= \Delta\Big[\left(\mathcal{O}_1^\dagger\right)\mathcal{H}^{(k+1)}\left(\mathcal{Z}_1^\dagger\right)\Big] \\ &= \left[\Delta\left(\mathcal{O}_1^\dagger\right)\right]\mathcal{H}^{(k+1)}\left(\mathcal{Z}_1^\dagger\right) \\ &\quad + \left(\mathcal{O}_1^\dagger\right)\left[\Delta\mathcal{H}^{(k+1)}\right]\left(\mathcal{Z}_1^\dagger\right) \\ &\quad + \left(\mathcal{O}_1^\dagger\right)\mathcal{H}^{(k+1)}\left[\Delta\left(\mathcal{Z}_1^\dagger\right)\right]. \end{aligned} \tag{19}$$

The uncertainty on the vectorized system matrix A is rewritten as

$$\begin{aligned} vec\Delta A &= \left(\left(\mathcal{Z}_1^{\dagger T}\mathcal{H}^{(k+1)T}\right)\otimes I_d\right) vec\left(\Delta\left(\mathcal{O}_1^\dagger\right)\right) \\ &\quad + \left(\mathcal{Z}_1^{\dagger T}\otimes\mathcal{O}_1^\dagger\right) vec\Delta\mathcal{H}^{(k+1)} \\ &\quad + \left(I_d\otimes\left(\mathcal{O}_1^\dagger\mathcal{H}^{(k+1)}\right)\right) vec\left(\Delta\left(\mathcal{Z}_1^\dagger\right)\right), \end{aligned} \tag{20}$$

where I_d is identity matrix with dimension d. $\otimes$ is the Kronecker product. The uncertainty of $\mathcal{H}^{(k+1)}$ can simply be estimated by cutting the signals.

The uncertainty on the pseudo-inverse of observability $\mathcal{O}_1$ can be defined directly from the singular values and singular vectors by

$$\begin{aligned} \Delta\left(\mathcal{O}_1^\dagger\right) &= \Delta\left(\Sigma_1^{-\frac{1}{2}}U_1^T\right) \\ &= \left[\Delta\left(\Sigma_1^{-\frac{1}{2}}\right)\right]U_1^T + \Sigma_1^{-\frac{1}{2}}\Delta\left(U_1^T\right) \\ &= -\frac{1}{2}\Sigma_1^{-\frac{3}{2}}\left(\Delta\Sigma_1\right)U_1^T + \Sigma_1^{-\frac{1}{2}}\Delta\left(U_1^T\right) \end{aligned} \tag{21}$$

The uncertainty of $\mathcal{O}_1^\dagger$ is now vectorized as

$$\begin{aligned} vec\left(\Delta\left(\mathcal{O}_1^\dagger\right)\right) &= \left(U_1\otimes\left(-\frac{1}{2}\Sigma_1^{-\frac{3}{2}}\right)\right) vec\Delta\Sigma_1 \\ &\quad + \left(I_{(p+1)r}\otimes\Sigma_1^{-\frac{1}{2}}\right) vec\left(\Delta\left(U_1^T\right)\right) \\ &= \left(U_1\otimes\left(-\frac{1}{2}\Sigma_1^{-\frac{3}{2}}\right)\right) vec\Delta\Sigma_1 \\ &\quad + \left(I_{(p+1)r}\otimes\Sigma_1^{-\frac{1}{2}}\right) P_{U_1}\ vec\Delta U_1 \end{aligned} \tag{22}$$

where P_{U_1} is a matrix that can permutate $vec\Delta U_1$ to $vec\left(\Delta\left(U_1^T\right)\right)$.

Similarly, the uncertainty on the pseudo-inverse of controllability $\mathcal{Z}_1$ can be descibed as

$$\begin{aligned}
\Delta\left(\mathcal{Z}_1^{\dagger}\right) &= \Delta\left(V_1\Sigma_1^{-\frac{1}{2}}\right) \\
&= (\Delta V_1)\,\Sigma_1^{-\frac{1}{2}} + V_1\Delta\left(\Sigma_1^{-\frac{1}{2}}\right) \\
&= (\Delta V_1)\,\Sigma_1^{-\frac{1}{2}} + V_1\left(-\frac{1}{2}\right)\Sigma_1^{-\frac{3}{2}}\Delta\Sigma_1 \qquad (23)
\end{aligned}$$

and reconstructing it in vectorized form leads to

$$\begin{aligned}
vec\left(\Delta\left(\mathcal{Z}_1^{\dagger}\right)\right) &= \left(\Sigma_1^{-\frac{1}{2}}\otimes I_{qr}\right) vec\Delta V_1 \\
&\quad + \left(I_d\otimes\left(-\frac{1}{2}V_1\Sigma_1^{-\frac{3}{2}}\right)\right) vec\Delta\Sigma_1. \qquad (24)
\end{aligned}$$

The sensitivity of the left singular vectors can be related to the uncertainty of subspace matrix $\mathcal{H}^{(k)}$ (see in [11])

$$vec\Delta U_1 = L_{1d}\begin{bmatrix} B_1^{\dagger}C_1 \\ \vdots \\ B_d^{\dagger}C_d \end{bmatrix} vec\Delta\mathcal{H}^{(k)} \qquad (25)$$

with a selection matrix defined by

$$L_{1d} = I_d\otimes\begin{bmatrix} I_{(p+1)r} & O_{(p+1)r\times qr} \end{bmatrix} \qquad (26)$$

and

$$B_j = \begin{bmatrix} I_{(p+1)r} & -\frac{\mathcal{H}^{(k)}}{\sigma_j} \\ -\frac{(\mathcal{H}^{(k)})^T}{\sigma_j} & I_{qr} \end{bmatrix}, \qquad (27)$$

$$C_j = \frac{1}{\sigma_j}\begin{bmatrix} v_j^T\otimes(I_{(p+1)r} - u_ju_j^T) \\ (u_j^T\otimes(I_{qr} - v_jv_j^T))\mathcal{P} \end{bmatrix}, \qquad (28)$$

$$\mathcal{P} = \sum_{k_1=1}^{(p+1)r}\sum_{k_2=1}^{qr} E_{k_1k_2}^{(p+1)r\times qr}\otimes E_{k_2k_1}^{qr\times(p+1)r}, \qquad (29)$$

where σ_j is the eigenvalue at system order j $\{j\in 1,\ldots,d\}$, u_j (resp. v_j) is column number j of U (resp. V). $E_{k_1k_2}^{(p+1)r\times qr}$ is a $(p+1)r\times qr$ matrix whose element is 1 at position (k_1,k_2) and zero elsewhere.

The sensitivity of eigenvalues is addressed as (see in [11])

$$vec(\Delta\Sigma_1) = S_{3d}\begin{bmatrix} (v_1\otimes u_1)^T \\ \vdots \\ (v_d\otimes u_d)^T \end{bmatrix} vec\Delta\mathcal{H}^{(k)}, \qquad (30)$$

in which S_{3d} is a selection matrix

$$S_{3d} = \sum_{s=1}^{d} E^{d^2 \times d}_{(s-1)d+s,s} \tag{31}$$

The sensitivity of right eigenvectors (see in [11]) is then specified by

$$vec\Delta V_1 = L_{2d} \begin{bmatrix} B_1^{\dagger} C_1 \\ \vdots \\ B_d^{\dagger} C_d \end{bmatrix} vec\Delta\mathcal{H}^{(k)} \tag{32}$$

with selection matrix

$$L_{2d} = I_d \otimes \begin{bmatrix} O_{qr\times(p+1)r} & I_{qr} \end{bmatrix}. \tag{33}$$

Especially, $vec\Delta\mathcal{H}^{(k)}$ can be simplified by making use of a block-storing matrix $M^{(k)}$

$$vec\Delta\mathcal{H}^{(k)} = S_4^{(k)} vec\Delta M^{(k)} \tag{34}$$

where

$$M^{(k)} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_{p+q+k} \end{bmatrix} \tag{35}$$

$$S_4^{(k)} = \begin{bmatrix} I_r \otimes S_{5,1}^{(k)} \\ I_r \otimes S_{5,2}^{(k)} \\ \vdots \\ I_r \otimes S_{5,q}^{(k)} \end{bmatrix} \tag{36}$$

$$S_{5,t}^{(k)} = \begin{bmatrix} O_{(p+1)r\times(t-1+k)r} & I_{(p+1)r} & O_{(p+1)r\times(q-t)r} \end{bmatrix} \tag{37}$$

Finally, the uncertainty of system matrix A can be shown in vectorization form

$$vec\Delta A = J_A \; vec\Delta M^{(k)}, \tag{38}$$

in which J_A is a Jacobian matrix

$$\begin{aligned} J_A = \; & N_1 \begin{bmatrix} (v_1 \otimes u_1)^T \\ \vdots \\ (v_d \otimes u_d)^T \end{bmatrix} S_4^{(k)} \\ & + N_2 \begin{bmatrix} B_1^{\dagger} C_1 \\ \vdots \\ B_d^{\dagger} C_d \end{bmatrix} S_4^{(k)} \\ & + \left(\mathcal{Z}_1^{\dagger T} \otimes \mathcal{O}_1^{\dagger} \right) S_4^{(k+1)} \end{aligned} \tag{39}$$

with the matrices

$$\begin{aligned} N_1 &= \left(\left({\mathcal{Z}_1^\dagger}^T {\mathcal{H}^{(k+1)}}^T\right) \otimes I_d\right) \left(U_1 \otimes \left(-\frac{1}{2}\Sigma_1^{-\frac{3}{2}}\right)\right) S_{3d} \\ &\quad + \left(I_d \otimes \left(\mathcal{O}_1^\dagger \mathcal{H}^{(k+1)}\right)\right) \left(I_d \otimes \left(-\frac{1}{2} V_1 \Sigma_1^{-\frac{3}{2}}\right)\right) S_{3d}, \end{aligned} \tag{40}$$

$$\begin{aligned} N_2 &= \left(\left({\mathcal{Z}_1^\dagger}^T {\mathcal{H}^{(k+1)}}^T\right) \otimes I_d\right) \left(I_{(p+1)r} \otimes \Sigma_1^{-\frac{1}{2}}\right) P_{U_1}\ L_{1d} \\ &\quad + \left(I_d \otimes \left(\mathcal{O}_1^\dagger \mathcal{H}^{(k+1)}\right)\right) \left(\Sigma_1^{-\frac{1}{2}} \otimes I_{qr}\right) L_{2d}. \end{aligned} \tag{41}$$

Likewise, the uncertainty of the vectorized system matrix C is

$$vec\Delta C \quad = \quad J_C\ vec\Delta M^{(k)} \tag{42}$$

with Jacobian matrix

$$J_C \quad = \quad (I_d \otimes S_C)\,(\mathcal{B}_d + \mathcal{C}_d)\, S_4^{(k)}, \tag{43}$$

where

$$S_C \quad = \quad \begin{bmatrix} I_r & O_{r\times pr} \end{bmatrix}, \tag{44}$$

$$\mathcal{B}_d \quad = \quad \left(I_d \otimes \left(\frac{1}{2} U_1 \Sigma_1^{-\frac{1}{2}}\right)\right) S_{3d} \begin{bmatrix} (v_1 \otimes u_1)^T \\ \vdots \\ (v_d \otimes u_d)^T \end{bmatrix} \tag{45}$$

$$\mathcal{C}_d \quad = \quad \left(\Sigma_1^{\frac{1}{2}} \otimes I_{(p+1)r}\right) L_{1d} \begin{bmatrix} B_1^\dagger C_1 \\ \vdots \\ B_d^\dagger C_d \end{bmatrix}. \tag{46}$$

Finally, the uncertainty of system matrices can be joined together

$$\begin{aligned} \begin{bmatrix} vec\Delta A \\ vec\Delta C \end{bmatrix} &= \begin{bmatrix} J_A \\ J_C \end{bmatrix} vec\Delta M^{(k)} \\ &= J_{A,C}\ vec\Delta M^{(k)} \end{aligned} \tag{47}$$

Then, the covariances of the modal parameters are obtained as

$$\begin{aligned} cov(f_\mu) &= J_{f_\mu}\ J_{A,C}\ cov(vec\ M^{(k)})\ J_{A,C}^T\ J_{f_\mu}^T \\ cov(d_\mu) &= J_{d_\mu}\ J_{A,C}\ cov(vec\ M^{(k)})\ J_{A,C}^T\ J_{d_\mu}^T \\ cov(\phi_\mu) &= J_{\phi_\mu}\ J_{A,C}\ cov(vec\ M^{(k)})\ J_{A,C}^T\ J_{\phi_\mu}^T \end{aligned} \tag{48}$$

4 Numerical Examples

The S101 bridge ([13]) connected Salzburg - Vienna carriage way in Austria. That is a post-tensioned concrete bridge with main span of 32 m, side spans of 12 m, and the width of 6.6 m. The deck is continuous over the piers. This bridge, contructed in 1960, has been a typical overpass bridge in Austria national highway. In the current paper, the ambient vibration data is collected on 15 sensors. The original sampling frequency is 500 Hz with 165000 time samples available. The data is decimated to 35.7 Hz and only five modes are taken into account.

Figure 1: S101 bridge in Reibersdorf, Austria

4.1 Modal analysis

For the output-only modal analysis of the ambient vibration data of the S101 bridge, similar parameters for both subspace algorithm and ERA are employed. 64 correlations ($p + 1 = q = 32$) are used, leading to Hankel matrices with 32 block rows and columns. The resulting multi-order diagrams are presented in Figure 2 and Figure 3.

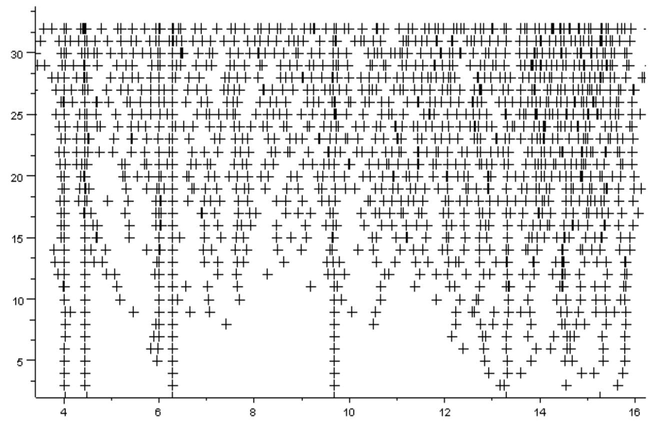

Figure 2: Stabilization diagram with subspace algorithm (natural frequency vs. model order)

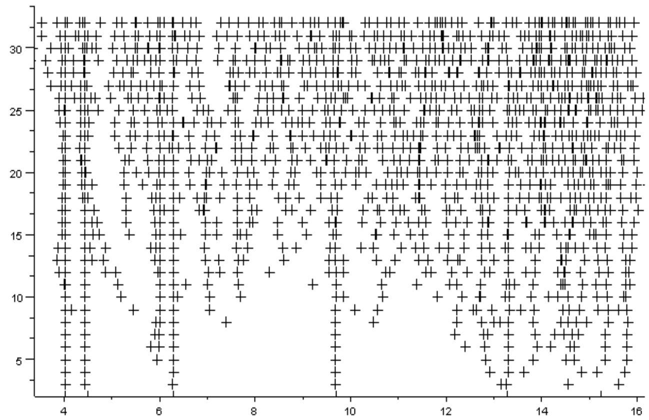

Figure 3: Stabilization diagram with ERA (natural frequency vs. model order)

The summary of the frequencies and damping ratios of the five identified modes is given in Table 1 and Table 2, for both subspace identification and ERA. ERA-fl is the ERA which uses first-lag and second-lag subspace matrices, ERA-sl is the ERA which utilizes second-lag and third-lag subspace matrices, ERA-tl is the ERA which employs third-lag and fourth-lag subspace matrices.

The differences in the obtained frequencies between subspace identification and ERA are small, less than 0.5%, for all five modes. In the case of damping ratios, the differences are bigger because of higher uncertainty in the estimation of damping ratios.

Mode	Frequency f (Hz)			
	Subspace	ERA-fl	ERA-sl	ERA-tl
1	4.039	4.038	4.037	4.037
2	6.282	6.283	6.284	6.283
3	9.682	9.684	9.683	9.684
4	13.284	13.283	13.290	13.307
5	15.721	15.720	15.763	15.630

Table 1: Identified frequencies with subspace algorithm and ERA

Mode	Damping Ratio d (%)			
	Subspace	ERA-fl	ERA-sl	ERA-tl
1	0.759	0.754	0.762	0.756
2	0.617	0.608	0.588	0.573
3	1.193	1.189	1.185	1.228
4	1.436	1.424	1.309	1.135
5	1.638	1.966	2.360	2.475

Table 2: Identified damping ratios with subspace algorithm and ERA

4.2 Confidence Intervals

For the computation of confidence intervals on modal parameters, 24 time lags, leading to $p+1 = q = 12$, and 40 model orders are utilized due to the limitation in computer memory.

Mode	Frequency confidence intervals (%)			
	Subspace	ERA-fl	ERA-sl	ERA-tl
1	0.115	0.110	0.122	0.100
2	0.092	0.093	0.092	0.089
3	0.133	0.134	0.156	0.159
4	0.471	0.300	0.247	0.573
5	1.148	1.825	2.485	1.442

Table 3: Frequency confidence intervals with subspace algorithm and ERA

Mode	Damping-ratio confidence intervals (%)			
	Subspace	ERA-fl	ERA-sl	ERA-tl
1	17.791	17.819	17.757	16.352
2	19.383	19.239	20.401	19.900
3	16.845	13.302	11.332	11.960
4	28.919	15.012	56.777	65.019
5	60.522	70.800	161.749	130.691

Table 4: Damping-ratio confidence intervals with subspace algorithm and ERA

In Table 3 and Table 4, the confidence intervals of the natural frequencies and damping ratios of the five modes are presented, respectively. Confidence intervals of the frequencies are much smaller than those of damping ratios. This is coherent with statistical theory, since the lower bound of the covariance given by Fisher information matrix is smaller for the frequencies than for the damping ratios. Besides, for this application, confidence bounds on modal parameters of ERA are relatively similar as those obtained with the subspace algorithm. While shifting the lags of ERA, the confidence interval fluctuations seem to be stable, and the ERA-fl may supply the most comparable results to subspace identification.

5 Conclusions

In this paper, the output-only system identification and confidence intervals on modal parameters are derived and implemented for both subspace algorithm and ERA. All the methods were successfully applied and tested on the ambient vibration data of the S101 overpass bridge.

The subspace algorithm and ERA give comparable results. The quality of stabilization diagrams as well as frequencies of subspace algorithm and ERA are almost similar. The damping ratios are slightly different due to an expectedly higher uncertainty on estimation.

The confidence intervals on modal parameters are also computed. It is observed that the uncertainty for ERA is relatively similar with that associated to subspace algorithm. When taking into account the lag effect for ERA, ERA-fl seems to be the most reasonable selection for the designers dealing with ERA.

Contents

References

[1] D. Bauer, M. Deistler, and W. Scherre. Consistency and asymtotic normality of some subspace algorithms for systems without observed inputs. *Automatica*, 35:1243–1254, 1999.

[2] D. Bauer and M. Jansson. Analysis of the asymtotic properties of the moesp type of subspace algorithms. *Automatica*, 36:497–509, 2000.

[3] A. Benveniste and J.-J. Fuchs. Single sample modal identification of a nonstationary stochastic process. *IEEE Transactions on Automatic Control*, AC-30(1):66–74, 1985.

[4] A. Benveniste and L. Mevel. Non-stationary consistency of subspace methods. *IEEE Transactions on Automatic Control*, AC-52(6):974–984, 2007.

[5] A. Chiuso and G. Picci. Asymptotic variance of subspace methods by data orthogonalization and model decoupling: a comparative analysis. *Automatica*, 40:1705–1717, 2004.

[6] M. Deistler, K. Peternell, and W. Scherrer. Consistency and relative efficiency of subspace methods. *Automatica*, 31:1865–1875, 1995.

[7] M. Döhler, X.-B. Lam, and L. Mevel. Confidence intervals on modal parameters in stochastic subspace identification. In *Proceedings of the 34th IABSE Symposium*, Venice, Italy, September 2010.

[8] J.-N. Juang and R. S. Pappa. An eigensystem realization algorithm for modal parameter identification and model reduction. *Journal of Guidance, Control and Dynamics*, 8(5):620–627, 1985.

[9] W. E. Larimore. System identification, reduced order filters and modelling via canonical variate analysis. In *the American Control Conference*, pages 445–451, 1983.

[10] B. Peeters and G. De Roeck. Reference-based stochastic subspace identification for output-only modal analysis. *Mechanical Systems and Signal Processing*, 13(6):855–878, November 1999.

[11] R. Pintelon, P. Guillaume, and J. Schoukens. Uncertainty calculation in (operational) modal analysis. *Mechanical Systems and Signal Processing*, 21:2359–2373, 2007.

[12] E. Reynders, R. Pintelon, and G. De Roeck. Uncertainty bounds on modal parameters obtained from stochastic subspace identification. *Mechanical Systems and Signal Processing*, 22(4):948–969, 2008.

[13] D. M. Siringoringo, T. Nagayama, Y. Fujino, D. Su, and C. Tandian. Observed dynamic characteristics of an overpass bridge during a full-scale destructive testing. In *The Fifth International Conference on Bridge Maintenance, Safety, Management and Life-Cycle Optimization*, 2010.

[14] P. Van Overschee and B. De Moor. *Subspace Identification for Linear Systems: Theory, Implementation, Applications*. Kluwer, 1996.

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